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MODEL THEORY VIA SET THEORY

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ABSTRACT

We develop a technique for applying models of set theory to obtain results in the model theory of infinitary languages. Some results on Δ-logics are also discussed.

0. Introduction

We develop a general technique for obtaining various classes of results in model theory by constructing certain models of fragments of true set theory.

In the first section, the necessary machinery is set up. In particular, formalisation of the metalanguage and of various logics is discussed. No new ideas are involved here, but we take the opportunity to establish our conventions and discuss some points which will be important later. We also list the results on models of set theory which we will need. Of these, only Theorem 1.5 is new, the others appear in [6], [7], and [12].

In the second section, the method is applied to obtain results on Hanf numbers, two cardinal theorems, undefinability of various well-orderings, compactness, and axiomatisability, for the logics $L_{\omega_1\omega}$, $L_{\mathscr{A}}$, LQ, etc. The results are not new, but the proofs are shorter than the standard ones, and indeed much more direct, modulo the results on models of set theory.

In the third section we take the opportunity to collect together a few results on Δ-logics, only one of which however is proved using models of set theory.

An important feature of the technique we use is that only certain set theoretic absoluteness conditions (as discussed in Section 1), in passing back and forth between the real world and models thereof, are used. Thus only minimal properties are required of the logics involved, and in many cases it is possible to single out larger classes of logics to which the results in Section 2 apply. However, we will not do this.

Except in the proof of Theorem 3.5, the absoluteness conditions used are

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essentially cardinality ones. By more delicate constructions, it is possible to keep certain classes of orderings absolute. In particular, building on the above mentioned theorem, J. Stavi and I have recently used these methods to obtain a compact axiomatisable extension of LQ by adding quantifiers corresponding to various stationary subsets of ω_1. Shelah [18], however, earlier obtained similar results using different techniques.

Finally, it should be mentioned that the technique gives a useful handle on certain Δ-logics.

I wish to thank my thesis advisor, H. Friedman, to whom this paper owes much. In particular, my first exposure to arguments of this type was Friedman's original proof of the failure of interpolation for $\Delta(LQ)$ in [5] (although there later proved to be a direct argument, cf. Theorem 3.1). The model of the real world which Friedman used was well-founded, but it turns out that the real power of the method lies in taking non-well-founded models. However, Friedman has also used non-well-founded models before in related situations, in particular in his proof of Lindstrom's characterisation of $L_{\omega\omega}$ [4].

1. Preliminaries

We will be discussing various languages, and logics associated with these languages. The metalogic in which we carry out these discussions, and the underlying metatheory, will be *ZFC*, Zermelo Fraenkel set theory with the axiom of choice. The techniques we develop rely heavily on the fact that the metalogic can be formalised, but we only actually carry out the formalisation so far as is necessary or illuminating.

The cumulative hierarchy of sets, $V = \langle V, \in \rangle$, is called the *real world* (of mathematics). To fix our ideas, it is helpful and indeed desirable to consider this model of the metatheory. A theorem in this paper, e.g. Theorem 2.1, is actually a metatheorem, or, as in the case of Theorem 1.1, a metatheorem schema (in both cases, after appropriate universal quantification). We may consider such a theorem as a result about (the members of) V. We assert something to be the case in V only if it is a provable consequence of *ZFC*. *Unless noted otherwise, all future discussions are carried out in the metalogic, and as such may be taken as statements concerning the members of* V. A *set* is a member of V, a *class* is a definable relation on V (definable in the metalogic, perhaps with parameters from V).

LANGUAGES AND LOGICS. A *language* is a set of relation, function, and constant symbols, such symbols being regarded as sets in the usual way. *Unless* *noted otherwise, all languages are subsets of HF* (the set of hereditarily finite sets), and hence countable. Languages will usually be written $L, K, \cdots$.

A structure for L, or *L-structure*, is a pair $\mathfrak{M} = \langle M, f\rangle$ where M is a non-empty set called the *universe* of $\mathfrak{M}$, and f is a map assigning to each member of L an interpretation in $\mathfrak{M}$ of the appropriate kind. $\mathfrak{M}$ will often be written as $\langle M, P^{\mathfrak{M}}, \cdots\rangle$ where $P^{\mathfrak{M}} = f(P)$, or even as $\langle M, P, \cdots\rangle$. Structures will generally be denoted $\mathfrak{M}, \mathfrak{N}, \mathfrak{R}, \cdots$, with universes $M, N, R, \cdots$ respectively.

A logic is a function * assigning to each language L a pair $\langle L^*, \models_L^*\rangle$, where L^* is a (possibly proper) class called the class of *L-sentences* of the logic, and $\models_L^*$ is a relation between the L-structures and L-sentences of the logic. We call $\models_L^*$ the *satisfaction relation* for L in the logic *, and usually write $\mathfrak{M} \models_L^* \varphi$ for $\langle M, \varphi\rangle \in \models_L^*$, where $\mathfrak{M}$ is an L-structure and φ an L-sentence of *. If Φ is a set of sentences, $\mathfrak{M} \models_L^* \Phi$ means $\forall \varphi \in \Phi(\mathfrak{M} \models_L^* \varphi)$. If no confusion arises, we suppress $\models_L^*$ and denote the logic by L^* (somewhat analogously to denoting a function f by $f(x)$). Instead of $\models_L^*$ we often write $\models^*$, $\models_L$, or even $\models$.

For particular logics it is usually more convenient to define first a notion of formula, such that every sentence is a formula (but not conversely), and to define satisfaction by induction on formula complexity. $\mathfrak{M} \models \varphi[a_1, \cdots, a_n]$ will then have its usual meaning. L^* will also be used to denote the set of formulae of the logic.

$L_{\infty\omega}$ is defined as usual. If κ is an infinite cardinal, $L_{\kappa\omega} = H(\kappa) \cap L_{\infty\omega}$ (where $H(\kappa) = \{x : |TC(x)| < \kappa\}$). $L_{\omega\omega}$ is often denoted L. $L_{\infty\omega}Q$ is defined as for $L_{\infty\omega}$ except that formulae of the form $Qv\varphi$, where v is a variable and φ is already a formula of $L_{\infty\omega}Q$, are allowed in the inductive definition. $Qv\varphi$ is interpreted as "there exist at least $\aleph_1$ many v such that φ". $L_{\kappa\omega}Q = L_{\infty\omega}Q \cap H(\kappa)$, $LQ = L_{\omega\omega}Q$. If $\mathscr{A} \supset HF$ is a transitive set, $L_{\mathscr{A}} = L_{\infty\omega} \cap \mathscr{A}$ and $L_{\mathscr{A}}Q = L_{\infty\omega}Q \cap \mathscr{A}$. If furthermore $L_{\mathscr{A}}$ is closed under subformulae, the finitary logical operations, and changes of variables, $L_{\mathscr{A}}$ is called a *fragment*.

THE LANGUAGE OF SET THEORY AND FORMALISING THE METALANGUAGE. We now discuss a particular language $S = \{\varepsilon\}$, the language of set theory, and its structures. Here ε is a binary relation symbol. Remember $S \in V$, S is not the metalanguage. S-structures will usually be written $\mathfrak{A} = \langle A, E\rangle$, $\mathfrak{B} = \langle B, F\rangle$, $\mathfrak{C} = \langle C, G\rangle, \cdots$. Every set A gives rise to the S-structure $\langle A, \in \restriction A\rangle$, which we usually just write as A.

Our S-structures will all satisfy some reasonable set theory. By *KP, Z, ZF* we mean the set of Kripke Platek, Zermelo, and Zermelo Fraenkel axioms respectively. If K is a set of axioms, KC is K augmented with the axiom of choice. From our point of view, $KP \in V$, $ZFC \in V$, etc.

If $\mathfrak{A}$ is an S-structure, by the *standard part* of $\mathfrak{A}$, sp$(\mathfrak{A})$, we mean the set of all $a \in A$ such that there is no infinite descending sequence $\cdots Ea_nE \cdots Ea_1Ea$. We often identify sp$(\mathfrak{A})$ with its transitive collapse. The ordinal of the standard part of $\mathfrak{A}$, osp$(\mathfrak{A})$, is the supremum of the ordinals in sp$(\mathfrak{A})$. If A is transitive, $\mathrm{o}(A)$ is the supremum of the ordinals in A. In this case $A = \mathrm{sp}(A)$ and $\mathrm{o}(A) = \mathrm{osp}(A)$.

The following is a useful bit of notation: if $a \in A$ then $a_E = \{b \in A : bEa\}$. Thus if $\mathfrak{A} = \langle A, \in \rangle$ then $a_E = a \cap A$. If $\mathfrak{A} = \langle A, E\rangle \subset \mathfrak{B} = \langle B, F\rangle$ and $a \in A$, then a is *fixed* (in passing from $\mathfrak{A}$ to $\mathfrak{B}$) if $a_E = a_F$, a is *enlarged* if $a_E \subsetneqq a_F$.

To each metaformula $P = P(x_1, \cdots, x_k)$ there is associated in a natural way an S-formula "P" $\in HF$, the *formal analogue* of P. If $\mathfrak{A} = \langle A, E\rangle$ is an S-structure, $a_1, \cdots, a_k \in A$, then

$$\mathfrak{A} \models \text{“}P\text{”}[a_1, \cdots, a_k] \quad \text{iff} \quad P^{\mathfrak{A}}(a_1, \cdots, a_k),$$

where $P^{\mathfrak{A}}$ is the relativisation of P to $\mathfrak{A}$. The proof is by induction on the complexity of P. If $\mathfrak{A} \models$ "P"$[a_1, \cdots, a_k]$ we say: $P(a_1, \cdots, a_k)$ in (or inside) $\mathfrak{A}$. By an abuse of language, we will refer to $P(a_1, \cdots, a_k)$ as a metaformula, and in the previous context write $\mathfrak{A} \models$ "$P(a_1, \cdots, a_k)$".

Suppose $\mathfrak{A} \models ZC$, say. By $\omega^{\mathfrak{A}}$ we mean the unique $a \in A$ such that $\mathfrak{A} \models$ "$a = \omega$". Similarly for $\omega_1^{\mathfrak{A}}$, $TC^{\mathfrak{A}}(x)$, $\langle xy \rangle^{\mathfrak{A}}$, etc. $a \in \in \mathfrak{A}$ will mean $TC(\{a\}) \subset A$ and $\langle TC(\{a\}), \in \rangle \cong \langle TC^{\mathfrak{A}}(\{a\}), E\rangle$ under the identity map.

LANGUAGES AND LOGICS WITHIN S-STRUCTURES. Suppose now $\mathfrak{A} \models ZC$. We can talk about the various languages, logics, satisfaction, etc., inside $\mathfrak{A}$. In the following, suppose the language $L \in \in \mathfrak{A}$. Then L is a language inside $\mathfrak{A}$ since the definition of being a language is given by a Σ_0 metaformula.

If the set $\mathfrak{M} \in A$ and $\mathfrak{M}$ is an L-structure in $\mathfrak{A}$, it will not necessarily be an L-structure in the real world (every structure is an ordered pair, $\mathfrak{M}$ may not even be an ordered pair in the real world). However, we can associate with $\mathfrak{M}$ a real world L-structure $\mathfrak{M}_E$ in a natural way. For suppose $\mathfrak{M} = \langle M, f\rangle$ inside $\mathfrak{A}$. Then by definition $\mathfrak{M}_E = \langle M_E, f^*\rangle$; where for constant symbols $c \in L$, $f^*(c) = a$ iff $f(c) = a$ in $\mathfrak{A}$; for function symbols $F \in L$, $f^*(F)$ $(a_1, \cdots, a_n) = a$ iff $f(F)$ $(a_1, \cdots, a_n) = a$ in $\mathfrak{A}$; and for relation symbols $R \in L$, $(a_1, \cdots, a_n) \in f^*(R)$ iff $(a_1, \cdots, a_n) \in f(R)$ in $\mathfrak{A}$. Although this definition disagrees with the previous definition for a_E where $a \in \mathfrak{A}$, there should be no confusion.

If $\mathfrak{A} = \langle A, E\rangle = \langle A, \in \rangle$, then $\mathfrak{M}_E = \mathfrak{M} \restriction A$, the restriction of $\mathfrak{M}$ to $M \cap A$.

Now suppose that φ is a formula of some logic *, and $\varphi \in \in \mathfrak{A}$. We will be interested in those $\mathfrak{A}$ such that $\mathfrak{A} \models$ "φ is a formula of *", and such that if, in $\mathfrak{A}$, $\mathfrak{M}$ is a structure for the language of φ, then for all $a_1, \cdots, a_n \in \mathfrak{M}_E$

$$\mathfrak{M} \models \varphi[a_1, \cdots, a_n] \quad \text{inside} \quad \mathfrak{A} \Leftrightarrow \mathfrak{M}_E \models \varphi[a_1, \cdots, a_n].$$

$\mathfrak{A} \models ZC$ is more than enough for the above to go through in case * is $L_{\infty\omega}$. If * is $L_{\omega_1\omega}$ then we need moreover that $\mathfrak{A} \models$ "$|TC(\varphi)| = \aleph_0$". If * is $L_{\omega_1\omega}Q$ or LQ, the notions of having cardinality $\geqq \aleph_1$ and cardinality $\leqq \aleph_0$ should be absolute between $\mathfrak{A}$ and V. In other words $|(\omega_1^{\mathfrak{A}})_E| \geqq \aleph_1$ and $|(\omega^{\mathfrak{A}})_E| = \aleph_0$; in particular $|(\omega_1^{\mathfrak{A}})_E| = \aleph_1$ and $|(\omega^{\mathfrak{A}})_E| = \aleph_0$ are enough. We discuss the restrictions on $\mathfrak{A}$ for other logics as the need arises.

CONSTRUCTING S-STRUCTURES. We turn to the problem of constructing various nice S-structures. Suppose $\Gamma = \{P_i : i = 1, \cdots, n\}$ where $P_i = P_i(x_1, \cdots x_k)$ are metaformulae. Then $\mathfrak{A} = \langle A, \in \rangle \prec_\Gamma V$ will mean

$$A \models ZC \;\&\; \bigwedge_{i=1}^{n} \left\{ \bigvee_{j=1}^{k} x_j \in A \, [P_i(x_1, \cdots, x_k) \Leftrightarrow A \models \text{"}P_i(x_1, \cdots, x_k)\text{"}] \right\}.$$

Such a are easily obtained from the following theorem. Γ will not normally be explicitly stated, but will be some finite set of metaformulae, sufficiently large to enable the argument in hand to carry through. We then write $A \prec V$. In particular, if $\mathfrak{M}$ is an L-structure we can choose Γ and A with $\mathfrak{M} \in A \prec V$ so that $L \in \in A$ and $A \models$ "$\mathfrak{M}$ is an L-structure".

THEOREM 1.1. *(Γ is a finite set of metaformulae.) Suppose X is a set. Then there exists $A \supset X$, $|A| = |X| + \aleph_0$, such that $A \prec_\Gamma V$.*

PROOF. From Levy [13, p. 48] we can find α so that $X \subset R_\alpha \prec_\Gamma V$. But $R_\alpha \models ZC$, and we can choose A so $X \subset A \prec_\Gamma R_\alpha$ and $|A| = |X| + \aleph_0$ by the downward Lowenheim Skolem theorem.

From the model $A = \langle A, \in \rangle$, we build non-standard models by means of the following theorems.

THEOREM 1.2. *Let $\mathfrak{A} = \langle A, E \rangle \models ZC$, $|A| = \aleph_0$, and c be a regular cardinal in $\mathfrak{A}$. Then there exists $\mathfrak{B} \succ \mathfrak{A}$ such that each $a \in c_E$ remains fixed, c is enlarged, and $|B| = \aleph_0$. Furthermore we may require either of the following two possibilities:*

*(i) there is no **least** new ordinal below c; or*

*(ii) (if $\mathfrak{A} \models$ "$c \neq \omega$") there is a **least** new ordinal below c.*

In the latter case, if we call the least new ordinal k, and if for some $s \in A$, $\mathfrak{A} \models$ "s is a stationary subset of c", we may also require that $\mathfrak{B} \models k\varepsilon s$.

The proof is in Hutchinson [7]. Whereas the result is stated there for $\mathfrak{A} \models ZFC$, as noted in [7] replacement is not necessary. For many applications we do not need (i) and (ii); the corresponding weaker result is proved in Keisler and Morley [12].

If we take $c = \omega_1^{\mathfrak{A}}$ and iterate Theorem 1.2 $\aleph_1$ times, then $|(\omega^{\mathfrak{B}})_F| = \aleph_0$ and $|(\omega_1^{\mathfrak{B}})_F| = \aleph_1$, giving the absoluteness conditions required for $L_{\omega_1\omega}Q$ and LQ. In more delicate applications, such as the construction mentioned in the Introduction of the compact extension of LQ, and in Section 3, cardinality absoluteness is not sufficient. We need to look at the possible order types of $(\omega_1^{\mathfrak{B}})_F$. These orderings are classified in [8].

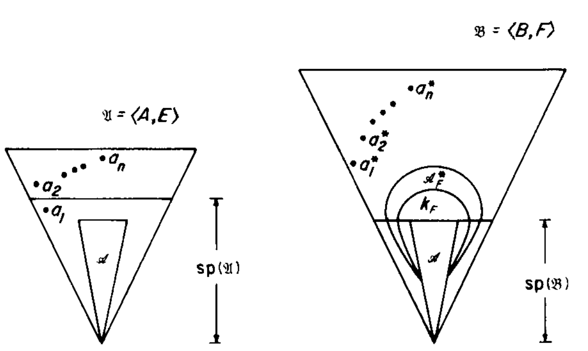
The next theorem is a consequence of theorems 4.4, 4.5, and the succeeding remarks, in [12].

THEOREM 1.3. *Let* $\mathfrak{A} = \langle A, E \rangle \models ZC$, $|A| = \aleph_0$, *c be a cardinal inside* $\mathfrak{A}$, *a be a regular cardinal inside* $\mathfrak{A}$, *and* $\mathfrak{A} \models$ "$b = \beth_a(c)$". *Let* $\langle X, < \rangle$ *be an arbitrary linear ordering. Then there exists* $\mathfrak{B} = \langle B, F \rangle > \mathfrak{A}$ *such that every member of* a_E *remains fixed,* $|c_F| = \aleph_0$, $\langle X, < \rangle \subset \langle b_F, F \rangle$, *and* $\langle X, < \rangle$ *is a set of indiscernibles for* $\langle \mathfrak{B}, a \rangle_{a \in A}$ *over* c_F. □

NOTE. In assuming $\mathfrak{A} \models$ "$b = {}^{\cdot}{}_a(c)$" we are supposing $\mathfrak{A}$ satisfies sufficient replacement for the Beth operation to be definable and have reasonable properties. A similar comment applies to Theorem 1.5.

REMARK. For the definition of indiscernibles *over* a subset of a structure, see Keisler [10], p. 88. From the proofs in [12], $\langle X, < \rangle$ can be seen to be a set of indiscernibles over c_F since if $d \in c_F$ and $d = t(x_1, \cdots, x_n)$ for $x_1 < \cdots < x_n$ in X, then $d = t(x_1', \cdots, x_n')$ for any $x_1' < \cdots < x_n'$ in X. Here t is a term in a certain expansion, by function symbols, of $S \cup \{a : a \in A\}$.

Theorem 1.4 and its proof are very similar to theorem 2.2 in Friedman [6]. Alternatively, as pointed out by the referee, the result follows easily from the pretty theorem 7 of Nadel [17]. It is the analogue of Theorem 1.2 (i), where instead of fixing the elements of c_E we fix the members of a countable admissible set $\mathscr{A}$. See the accompanying diagram. Admissible sets are discussed in Keisler [10].



THEOREM 1.4. *Let* $\mathscr{A}$ *be a countable admissible set,* $\mathfrak{A} = \langle a, E\rangle \models ZC$, $\mathscr{A} \in\in \mathfrak{A}$, *and* $a_1, \cdots, a_n \in A$. *Let* S' *be the extension of the language* S *of set theory obtained by adding constant symbols* $\bar{\mathscr{A}}, \bar{a}_1, \cdots, \bar{a}_n, \{\bar{a} : a \in \mathscr{A}\}$ *(where the map* $a \mapsto \bar{a}$ *is recursive over* $\mathscr{A}$*). Let* T *be a theory in* $S'_{\mathscr{A}}$ *which is* Σ_1*-definable over* $\mathscr{A}$. *Suppose* $\langle \mathfrak{A}, \mathscr{A}, a_1, \cdots, a_n, a\rangle_{a\in\mathscr{A}} \models T$.

Then there exists $\mathfrak{B} = \langle B, F\rangle \models ZC$ *such that* $|B| = \aleph_0$, $a \in\in \mathfrak{B}$ *if* $a \in \mathscr{A}$, $\mathrm{o}(\mathscr{A}) = \mathrm{osp}(\mathfrak{B})$; *and there exist* $\mathscr{A}^*, a_1^*, \cdots, a_n^* \in B$ *(*x^* *is the intended interpretation of* $\bar{x}$*) such that* $\langle \mathfrak{B}, \mathscr{A}^*, a_1^*, \cdots, a_n^*, a\rangle_{a\in\mathscr{A}} \models T$. *Furthermore, we may suppose that* $\langle \mathscr{A}^*_F, F\rangle$ *is an end extension of* $\langle k_F, F\rangle$ *which is an end extension of* $\langle \mathscr{A}, \in\rangle$, *for some* $k \in \mathscr{A}^*_F$.

PROOF. Let $S'' = S' \cup \{\bar{k}\}$, where $\bar{k}$ is a new individual constant with intended interpretation k. Let T' be the theory in $S''_{\mathscr{A}}$ with axioms:

$$T + ZC;$$

$$\forall v(v \varepsilon \bar{a} \leftrightarrow \bigvee\!\!\!\bigvee_{b\in a} v = \bar{b}), \quad \text{each} \quad a \in \mathscr{A};$$

$$\bar{a}\, \varepsilon\, \bar{k}, \qquad \text{each} \quad a \in \mathscr{A};$$

$$\bar{k}\, \varepsilon\, \bar{\mathscr{A}};$$

$$\bar{k} \text{ is transitive};$$

$$\bar{\mathscr{A}} \text{ is transitive}.$$

T' is consistent by Barwise compactness. By theorem 7 of [17] there is an admissible set $\mathscr{B} \supset \mathscr{A}$, having the same ordinals as $\mathscr{A}$, which contains a model $\mathfrak{B} = \langle B, F\rangle$ of T'. All that remains to be checked is that $\mathrm{o}(\mathscr{A}) = \mathrm{osp}(\mathfrak{B})$. But $\mathscr{A} \subset \mathrm{B}$ implies $\mathrm{o}(\mathscr{A}) \leqq \mathrm{osp}(\mathfrak{B})$, and $\mathfrak{B} \in \mathscr{B}$ implies $\mathrm{osp}(\mathfrak{B}) \leqq \mathrm{o}(\mathscr{B}) = \mathrm{o}(\mathscr{A})$.

REMARK. The theorem is usually applied in the following way. $\mathscr{A}$ will be a countable admissible set, and a finite number of metaformulae $P(\mathscr{A}, a_1, \cdots, a_n)$ will hold in the real world. Taking $\mathscr{A} \in\in A$, $a_1, \cdots, a_n \in A$, $A \prec_K V$ for sufficiently large K, and then applying the theorem, we obtain $\mathfrak{B}$ such that each $P(\mathscr{A}^*, a_1^*, \cdots, a_n^*)$ holds in $\mathfrak{B}$.

In particular, we may suppose $\mathfrak{B} \models$ "$\mathscr{A}^*$ is admissible". Now suppose some $R \subset \mathscr{A}$ is Σ_1-definable over $\mathscr{A}$ (usually R will be a set of sentences in some logic). Thus $R = \{x \in \mathscr{A} : \mathscr{A} \models \text{"}Q(x, b_1, \cdots, b_m)\text{"}\}$, where Q is a Σ_1-metaformula and $b_1, \cdots, b_m \in \mathscr{A}$. Then $R \in A$ since A satisfies separation, and so $R^* \in \mathfrak{B}$ where inside $\mathfrak{B}$, $R^* = \{x\, \varepsilon\, \mathscr{A}^* \models \text{"}Q(x, b_1, \cdots, b_m)\text{"}\}$. Moreover, working in $\mathfrak{B}$, if $r =$

$\{x \varepsilon k : k \models \text{“}Q(x, b_1, \cdots, b_m)\text{”}\}$, then $r \varepsilon \mathscr{A}^*$ since $\mathscr{A}^*$ is admissible and r is a Δ_0-definable over $\mathscr{A}^*$ subset of $k \varepsilon \mathscr{A}^*$. Also since $\langle \mathscr{A}^*_F, F\rangle$ is an end extension of $\langle k_F, F\rangle$ which is an end extension of $\langle \mathscr{A}, \in\rangle$, and Q is Σ_1, it follows $R \subset r_F \subset R^*_F$. In conclusion, there is an $r \in A^*_F$ such that $R \subset r_F \subset R^*_F$.

We now come to the analogue of Theorem 1.3 in which every member of some countable admissible $\mathscr{A}$ remains fixed.

THEOREM 1.5. *Let $\mathscr{A} \neq HF$ be a countable admissible set, $\mathfrak{A} = \langle A, E\rangle \models ZC$, $\mathscr{A} \in \in \mathfrak{A}$, and $a_1, \cdots, a_n \in \mathfrak{A}$. Let $\alpha = \mathrm{o}(\mathscr{A})$, c be a cardinal inside $\mathfrak{A}$, and $b = \beth_\alpha(c)$ inside $\mathfrak{A}$. Let S' be the extension of the language S of set theory obtained by adding constant symbols $\bar{\mathscr{A}}$, $\bar{\alpha}$, $\bar{a}_1, \cdots, \bar{a}_n$, $\bar{b}$, $\bar{c}$, $\{\bar{a} : a \in \mathscr{A}\}$ (where the map $a \mapsto \bar{a}$ is recursive over $\mathscr{A}$). Let T be a theory in $S'_{\mathscr{A}}$ which is Σ_1-definable over $\mathscr{A}$, and suppose $\langle \mathfrak{A}, \mathscr{A}, \alpha, a_1, \cdots, a_n, b, c, a\rangle_{a\in\mathscr{A}} \models T$. Let $\langle X, <\rangle$ be an arbitrary linear ordering.*

Then there exists $\mathfrak{B} = \langle B, F\rangle \models ZC$ such that $a \in \in \mathfrak{B}$ if $a \in \mathscr{A}$. Moreover, there exist $\mathscr{A}^*, \alpha^*, a_1^*, \cdots, a_n^*, b^*, c^* \in B$ (x^* is the intended interpretation of $\bar{x}$) such that $|c_F| = \aleph_0$ and $\mathfrak{B}' = \langle \mathfrak{B}, \mathscr{A}^*, \alpha^*, a_1^*, \cdots, a_n^*, b^*, c^*, a\rangle_{a\in\mathscr{A}} \models T + \bar{\alpha} = \mathrm{o}(\bar{\mathscr{A}}) + \bar{b} = \beth_{\bar{\alpha}}(\bar{c})$. Furthermore $\langle X, <\rangle \subset \langle b_F, F\rangle$, and $\langle X, <\rangle$ is a set of indiscernibles in $\mathfrak{B}'$ over c_F with respect to $S'_{\mathscr{A}}$.

PROOF. The easiest method is to first apply Theorem 1.4. The hypotheses remain unchanged, except that we now need to distinguish between $\mathscr{A}$ and the interpretation $\mathscr{A}^0$ of $\bar{\mathscr{A}}$. Similarly α^0 will interpret $\bar{\alpha}$. Furthermore, $\mathrm{osp}(\mathfrak{A}) = \alpha$ and $a \in \in \mathfrak{A}$ for each $a \in \mathscr{A}$.

We will just sketch the rest of the proof. Extend $S'_{\mathscr{A}}$ to a Skolem fragment $S^*_{\mathscr{A}}$ (as in [10], p. 67). We cannot pick definable Skolem functions in $\mathfrak{A}$, but via the axiom of choice in $\mathfrak{A}$ we can pick definable functions f_φ satisfying

$$(1) \qquad \forall v_1 \varepsilon b, \cdots, v_n \varepsilon b (\exists v \varphi(v, v_1, \cdots, v_n) \to \varphi(f_\varphi(v_1, \cdots, v_n), v_1, \cdots, v_n)),$$

for each $\varphi = \varphi(v, v_1, \cdots, v_n) \in S^*_{\mathscr{A}}$.

Let $\{\varphi_k : k < \omega\}$ be some enumeration of the formulae of $S^*_{\mathscr{A}}$. In the usual way we proceed by induction along this enumeration building a set T' of sentences of the form $\varphi(x_1, \cdots, x_n)$ where $x_1, \cdots, x_n \in X$ and $\varphi(v_1, \cdots, v_n) \in S^*_{\mathscr{A}}$. At each step in the procedure there will be some definable (in $\mathfrak{A}$) $I \subset b_F$, such that $|I| \geqq \beth_\beta(c)$ inside $\mathfrak{A}$, for some *non-standard* $\beta < \alpha^0$. Furthermore, each $\varphi(x_1, \cdots, x_n)$ so far obtained, where $x_1 < \cdots < x_n$, will hold in $\mathfrak{A}$ for $x_1, \cdots, x_n$ interpreted by any increasing n-tuple of members from I.

We can ensure that

(i) $$\Big\{ZC + T + \bar{\alpha} = o(\mathscr{A}) + \bar{b} = \beth_{\bar{\alpha}}(\bar{c}) + \Big\{\forall v(v\varepsilon\bar{a} \leftrightarrow \bigvee_{b\in a} v = \bar{b}) : a \in \mathscr{A}\Big\} + (1)\Big\} \subset T';$$

(ii) $$(\varphi_k(x_1,\cdots,x_n) \leftrightarrow \varphi_k(x'_1,\cdots,x'_n)) \in T';$$

where $x_1 < \cdots < x_n, x'_1 < \cdots < x'_n$;

(iii) $$\varphi_k(x_1,\cdots,x_n) \in T' \quad \text{or} \quad \neg\varphi_k(x_1,\cdots,x_n) \in T';$$

(iv) $$\bigvee \Phi(x_1,\cdots,x_n) \in T' \Rightarrow \varphi(x_1,\cdots,x_n) \in T' \quad \text{for some} \quad \varphi \in \Phi;$$

(v) $$(t(x_1,\cdots,x_n)\varepsilon\bar{c}) \in T' \Rightarrow (t(x_1,\cdots,x_n) = t(x'_1,\cdots,x'_n)) \in T',$$

where $x_1 < \cdots < x_n, x'_1 < \cdots < x'_n$,

and $t(v_1,\cdots,v_n)$ is a term in $S^*_{\mathscr{A}}$.

The argument of course uses the Erdos-Rado Theorem. For any two ordinals $\beta, \gamma < \alpha^0$, write $\gamma \ll \beta$ if $\gamma + \nu < \beta$ for all standard ν. Then for any non-standard β there is a non-standard $\gamma \ll \beta$, and if $\gamma \ll \beta$ then $\beth_\beta(c) \to (\beth_\gamma(c))^n_c$ is an easy consequence of the Erdos-Rado Theorem.

Build a model of T' from terms of $S^*_{\mathscr{A}}$ applied to X. Restricting to S' we have the required structure. In particular, that $|c_F| = \aleph_0$ and $\langle X, < \rangle$ is a set of indiscernibles over c_F with respect to $S'_{\mathscr{A}}$, is a consequence of the fact that if $t(x_1,\cdots,x_n) = d \in c_F$ where $x_1 < \cdots < x_n$, then $t(x'_1,\cdots,x'_n) = d$ for any $x'_1 < \cdots < x'_n$. □

REMARKS. (a) The trick of first making $\mathscr{A}$ non-standard is due to Friedman. Our original proof was much longer.

(b) Suppose $R \subset \mathscr{A}$ is Σ_1-definable over $\mathscr{A}$. Then as in the Remark following Theorem 1.4, we may suppose $R \subset R^*_F$. In fact, by a slight modification of the proof of Theorem 1.5, we may even suppose $R \subset r_F \subset R^*_F$ for some $r \in \mathscr{A}^*_F$.

2. Results in model theory

We will see how many classical model theoretic results are almost immediate consequences of the existence of various models of set theory. Standard notation is used; [10], henceforth referred to as I.L., and [16], are references. *Recall our conventions regarding $A < V$ as stated immediately preceding Theorem 1.1.*

The following two-cardinal theorem is due to Keisler, I.L., p. 116, and is the natural extension to $L_{\omega_1\omega}$ of a result of Vaught for $L_{\omega\omega}$. Notice that our proof does not require the rather complicated theorem 28 from p. 111 of I.L. The role of R in theorem 28 is, in a certain sense, taken over by the well-founded relation $\in$.

THEOREM 2.1. *Let $\mathfrak{M} = \langle M, U, \cdots \rangle$ be an L-structure with $|M| = \kappa$, $|U| = \lambda$, $\kappa > \lambda \geqq \aleph_0$. Let $L_{\mathscr{A}}$ be a countable fragment of $L_{\omega_1\omega}$. Then there exist structures $\mathfrak{N}$, $\mathfrak{R}$ such that $\mathfrak{N} <_{L_{\mathscr{A}}} \mathfrak{M}$, $\mathfrak{N} <_{L_{\mathscr{A}}} \mathfrak{R}$, $|\mathfrak{N}| = \aleph_0$, $|\mathfrak{R}| = \aleph_1$, and $U^{\mathfrak{N}} = U^{\mathfrak{R}}$. In particular, if T is a countable theory in $L_{\omega_1\omega}$ and T admits (κ, λ) for some infinite cardinals $\kappa > \lambda$, then T admits (ω_1, ω).*

PROOF. Take $A < V$ where $\mathscr{A} \subset A$, $\mathfrak{M}, \kappa, \lambda \in A$, and $|A| = \aleph_0$. By $\aleph_1$ applications of Theorem 1.2 take $\mathfrak{B} = \langle B, F \rangle > A$ such that λ remains fixed, $|(\lambda^+)_F| = \aleph_1$, $|B| = \aleph_1$. Let $\mathfrak{N} = \mathfrak{M} \restriction A$ and $\mathfrak{R} = \mathfrak{M}_F$.

It is easy to check that all the required conditions hold. However, as an illustration, we do it this time in detail.

Suppose $\varphi = \varphi(x_1, \cdots, x_n) \in L_{\mathscr{A}}$, $a_1, \cdots, a_n \in M \cap A$, and $\mathfrak{M} \models \varphi[a_1, \cdots, a_n]$. Then $\mathfrak{M} \models \varphi[a_1, \cdots, a_n]$ inside $\langle A, \in \rangle$ since $A < V$. Since $\varphi \in \mathscr{A}$, $\mathscr{A}$ is closed under transitive closure, and $\mathscr{A} \subset A$, it follows $\varphi \in \in A$. Hence $\mathfrak{M} \restriction A \models \varphi[a_1, \cdots, a_n]$ and so $\mathfrak{N} = \mathfrak{M} \restriction A <_{L_{\mathscr{A}}} \mathfrak{M}$.

Any $a \in \mathscr{A}$ is countable, so is countable in $\langle A, \in \rangle$. But $\lambda \geqq \aleph_0$, and anything in A of cardinality $\leqq \lambda$ in A remains fixed in passing to $\mathfrak{B}$. In particular $a \in \mathscr{A}$ remains fixed and so $\varphi \in \in \mathfrak{B}$. It now follows $\mathfrak{N} <_{L_A} \mathfrak{R}$ by a similar argument to before. Notice that $A \models ZC$ justifies our applying Theorem 1.2.

U has cardinality λ in A, so remains fixed in passing to $\mathfrak{B}$. In other words $U \cap A = U_F$, i.e. $U^{\mathfrak{N}} = U^{\mathfrak{R}}$, and both have cardinality $\aleph_0$. Clearly $|\mathfrak{N}| = \aleph_0$, and $|\mathfrak{R}| = \aleph_1$ since $|(\lambda^+)_F| = |B| = \aleph_1$. □

We next show that the Hanf number of $L_{\omega_1\omega}$ is $\leqq \beth_{\omega_1}$. By the usual example it is then precisely $\beth_{\omega_1}$. See I.L. p. 69, p. 78.

THEOREM 2.2. *Let $L_{\mathscr{A}}$ be a countable fragment of $L_{\omega_1\omega}$, T a set of sentences in $L_{\mathscr{A}}$. Suppose that T has models of power $\beth_\alpha$, for all $\alpha < \omega_1$. Then:*

(i) T has a model which has an infinite set of indiscernibles in $L_{\mathscr{A}}$;

(ii) T has models in all infinite powers.

PROOF. Take $A < V$ such that $\mathscr{A} \subset A$, $\mathscr{A}, T \in A$, and $|A| = \aleph_0$. Using Theorem 1.3 with $c = \omega$ and $a = \omega_1$, take $\mathfrak{B} = \langle B, F \rangle > A$ such that each member of $\omega_1 \cap A$ remains fixed, $\langle X, < \rangle \subset (\beth_{\omega_1})_F, < \rangle$, and $\langle X, < \rangle$ is a set of indiscernibles for $\mathfrak{B}$, where $\langle X, < \rangle$ is an arbitrary linear ordering.

Inside $\mathfrak{B}$, there exist models of T of arbitrarily large cardinality below $\beth_{\omega_1}$. Select X with some proper initial segment Y of arbitrary cardinality, and take

$b \in B$ with $Y < b < \beth_{\omega_1}$. Inside $\mathfrak{B}$ select $\mathfrak{M} \models T$ so that $b \subset M$. Then $\mathfrak{M}_F \models T$ and has cardinality $\geqq |Y|$. By downward Lowenheim Skolem theorem there is a model of cardinality precisely $|Y|$. This proves (ii).

To prove (i), we need to further stipulate that $\langle X, < \rangle$ is a set of indiscernibles for $\langle \mathfrak{B}, a \rangle_{a \in A}$ over $(\omega_1)_F$. Theorem 1.9 certainly allows this. Notice that $\beth_{\omega_1} = \beth_{\omega_1}(\beth_1)$, and $\beth_1 \geqq \omega_1$. Now there is in the real world a function g with domain ω_1, such that each $g(\alpha)$ is a model $\mathfrak{M}_\alpha \models T$ where the universe of $\mathfrak{M}_\alpha$ is $\beth_\alpha$. Then g has the same property in $\mathfrak{B}$. It follows that if $|Y| \geqq \aleph_0$ and in $\mathfrak{B}$, $g(b) = \mathfrak{M}$, then $\mathfrak{M}_F \models T$ and Y is a set of indiscernibles for $\mathfrak{M}_F$ in $L_{\mathscr{A}}$. □

REMARK. There is a direct argument from (i) to (ii) as in I.L. p. 78.

Theorem 31, I.L. p. 117, on homogeneous structures, theorem 12 (i), p. 49 on the undefinability in $L_{\omega_1\omega}$ of the class of well-orderings, theorem 23, p. 88, Morley's two-cardinal theorem for $L_{\omega_1\omega}$, and the results in [9], can easily be proved in a similar manner.

Next we come to logics over countable admissible sets. Theorem 2.3 is theorem 12 (ii), I.L. p. 49.

THEOREM 2.3. *Let $\mathscr{A} \subset HC$ be a countable admissible set. Let the language L contain a unary relation symbol U and binary relation symbol $<$. Let Φ be a theory in $L_{\mathscr{A}}$ which is Σ_1-definable over $\mathscr{A}$. Suppose that for all $\alpha < o(\mathscr{A})$, Φ has a model $\mathfrak{M} = \langle M, U, <, \cdots \rangle$ such that $<$ is a linear ordering of U and $\langle \alpha, < \rangle \subset \langle U, < \rangle$. Then Φ has a model containing a copy of the rationals.*

PROOF. Take $A \prec V$ such that $\mathscr{A} \subset A$, and $\mathscr{A}, L, \Phi \in A$. Then $\mathscr{A} \in \in A$ and $A \models ZC$.

Apply Theorem 1.4 and the following remark to $\langle A, \in \rangle$ to obtain $\mathfrak{B}, \mathscr{A}^*, L^*, \Phi^*$. We may suppose that, inside $\mathfrak{B}$,

(1) $\forall \alpha < o(\mathscr{A}^*) \exists \mathfrak{M} = \langle M, U, <, \cdots \rangle \forall \varphi \in \Phi^*$
$(\mathfrak{M} \models \varphi \wedge <$ is a l.o. of $U \wedge \langle \alpha, < \rangle \subset \langle U, < \rangle)$.

Select a non-standard b for α in (1), and let $\mathfrak{M}$ be the L^*-structure asserted to exist in $\mathfrak{B}$, so that $\mathfrak{M} \models \Phi^*$ in $\mathfrak{B}$. But $\Phi \subset \Phi^*_F$, and $\varphi \in \in \mathfrak{B}$ for each $\varphi \in \Phi$, hence $M_F \restriction L \models \Phi$ and contains a copy of the rationals. □

The following two-cardinal theorem for $L_{\mathscr{A}}$, $\mathscr{A}$ countable admissible, is due to Barwise and Kunen [2]. They actually have a result which holds for arbitrary (not necessarily countable) admissible $\mathscr{A}$.

THEOREM 2.4. *Let $\mathscr{A} \neq HF$ be a countable admissible set. Let L be a language containing the unary relation symbol U, and Φ a theory in $L_{\mathscr{A}}$ which is*

Σ_1-definable over $\mathscr{A}$. Suppose that for all $\alpha < \mathrm{o}(\mathscr{A})$ there exists a cardinal $\kappa \geqq \omega$ such that Φ admits $(\beth_\alpha(\kappa), \kappa)$. Then:
(i) *Φ has a model $\mathfrak{M} = \langle M, U, \cdots \rangle$ such that $|U| = \aleph_0$ and there is an infinite set of indiscernibles over U in $L_{\mathscr{A}}$;*
(ii) *For all cardinals $\lambda \geqq \omega$, Φ admits (λ, ω).*

PROOF. Select f so Φ admits $(\beth_\alpha(f(\alpha)), f(\alpha))$ for all $\alpha < \gamma = \mathrm{o}(\mathscr{A})$. Take a cofinal subsequence $\langle \beta_\alpha \rangle$ of $\mathrm{o}(\mathscr{A})$ so that $f(\beta_\alpha)$ is non-decreasing, and let $\kappa = \sup_\alpha \{f(\beta_\alpha)\}$. Take $A < V$ with $\mathscr{A} \subset A$, $\mathscr{A}, T, f, \langle \beta_\alpha \rangle$, $\kappa \in A$, and $|A| = \aleph_0$. Then $\mathscr{A} \in \in A$ and $A \models ZC$. Use Theorem 1.5 to construct $\mathfrak{B} = \langle B, F \rangle$ where $c = \kappa$. Thus we may suppose $|\kappa_F| = \aleph_0$, $X \subset (\beth_\gamma(\kappa))_F$ for $X = \langle X, < \rangle$ an arbitrary linear ordering, and indeed that X is a set of indiscernibles over κ_F.

In the real world and hence in $\mathfrak{B}$, T admits the pair $(\beth_{\beta_\alpha}(f(\beta_\alpha)), f(\beta_\alpha))$ for each α. Select X with an initial segment Y where $|Y| = \lambda$, and take $b \in B$ with $Y < b < \beth_\gamma(\kappa)$. Inside $\mathfrak{B}$ select $\mathfrak{M} \models T$ so that $b \subset M$ and $U^{\mathfrak{M}} \subset \kappa$. Then in the real world $\mathfrak{M}_F \models T$, $|M_F| \geqq |Y|$, and $|U^{\mathfrak{M}_F}| = \aleph_0$. By the downward Lowenheim Skolem theorem (ii) is established.

To prove (i) we need only note that every formula in $L_{\mathscr{A}}$ may be coded as an integer and so certainly as a member of κ, and that this then holds in $\mathfrak{B}$. Thus the fact that $\langle Y, < \rangle$ is a set of indiscernibles in $\mathfrak{B}$ (strictly, $\mathfrak{B}$ augmented with certain constant symbols) over κ_F implies that $\langle Y, < \rangle$ is a set of indiscernibles in $\mathfrak{M}_F$ over U with respect to the logic $L_{\mathscr{A}}$. □

That the Hanf number of $L_{\mathscr{A}}$ is $\beth_{\mathrm{o}(\mathscr{A})}$, for $\mathscr{A}$ countable admissible, follows easily from the above result. Alternatively, it may be proved directly as above, although a little more quickly.

Compactness and axiomatisability for LQ and $L_{\mathscr{A}}Q$ can also be shown by our methods. We give two examples.

THEOREM 2.5. *The set of universally valid sentences in LQ is recursively enumerable.*

PROOF. Suppose φ is a sentence of LQ. Then we claim

(1) $$\varphi \text{ is valid} \Leftrightarrow ZC \models \text{“}\varphi \text{ is valid”}.$$

Remember that $\varphi \in HF$, and "φ is valid" is an S-sentence.

Suppose then that φ is not valid, hence $\neg\varphi$ has a model, and so taking $A < V$, we have $\langle A, \in \rangle \models$ "$\neg\varphi$ has a model" and so $\langle A, \in \rangle \models \neg$"$\varphi$ is valid" But $A \models ZC$ and so $ZC \not\models$ "φ is valid".

Conversely, if $ZC \not\models$ "φ is valid", then there is a countable $\mathfrak{A} =$

$\langle A, E\rangle \models ZC +$ "$\neg\varphi$ has a model". Apply Theorem 1.2 $\aleph_1$ times to construct $\mathfrak{B} = \langle B, F\rangle > \mathfrak{A}$ with $|(\omega_F^{\mathfrak{B}})| = \aleph_0$, $|(\omega_1^{\mathfrak{B}})_F| = \aleph_1$. Then in $\mathfrak{B}$ there is a model $\mathfrak{M} \models \neg\varphi$, and so $\mathfrak{M}_F \models \neg\varphi$, thus φ is not valid.

Thus

$$\varphi \text{ is valid} \Leftrightarrow \exists x \in HF \qquad (x = \text{“}\varphi \text{ is valid”} \wedge ZC \vdash x).$$

By the Godel completeness theorem for S, it follows that the set of valid sentences of LQ is recursively enumerable. □

The next result, and also completeness for $L_{\mathscr{A}}Q$, are both due to Barwise (unpublished) and to Keisler [11], independently.

Assume in the theorem that $\mathscr{A}$ is a countable admissible set and the language $L \subset \mathscr{A}$ is a Σ_1-definable set over $\mathscr{A}$.

THEOREM 2.6. *$L_{\mathscr{A}}Q$ is compact in the Barwise sense.*

PROOF. Suppose Φ is a set of sentences of $L_{\mathscr{A}}Q$ which is Σ_1-definable over $\mathscr{A}$, and suppose every $\mathscr{A}$-finite subset of Φ has a model (x is $\mathscr{A}$-finite if and only if $x \in \mathscr{A}$). We wish to show Φ has a model. Take $A \prec V$ such that $\mathscr{A} \subset A$, $\mathscr{A} \in A$. Construct $\mathfrak{B}$ from $\langle A, \in\rangle$ as in Theorem 1.4. Construct $\mathfrak{C} = \langle C, G\rangle > \mathfrak{B}$ by Theorem 1.2 so that $\omega = \omega^{\mathfrak{B}}$ remains fixed, and $|(\omega_1^{\mathfrak{C}})_G| = \aleph_1$.

Let $\mathscr{A}^*, L^*, \Phi^*$ be the interpretations in $\mathfrak{B}$, and hence in $\mathfrak{C}$, of $\mathscr{A}, L, \Phi$ respectively. Since $\mathscr{A}, L, \Phi$ are countable in V, hence in A, we may suppose $\mathscr{A}^*, L^*, \Phi^*$ are countable in $\mathfrak{B}$, and hence fixed in passing to $\mathfrak{C}$.

We may assume that in $\mathfrak{C}$ every $\mathscr{A}^*$-finite subset of Φ^* has a model. By the Remark following Theorem 1.4, there is a $\varphi \in \mathscr{A}_F^* = \mathscr{A}_G^*$ such that $\varphi \subset \varphi_F (= \varphi_G) \subset \Phi_F^* (= \Phi_G^*)$. In $\mathfrak{C}, \varphi$ has a model $\mathfrak{M}$, and so $\mathfrak{M}_G \restriction L \models \Phi$, since $|(\omega^{\mathfrak{C}})_G| = \aleph_0$ and $|(\omega_1^{\mathfrak{C}})_G| = \aleph_1$. □

3. Δ-Logics

The sigma logic, $\Sigma(L^*)$, is defined as follows. Sentences of $\Sigma(L^*)$, in the language L, are objects of the form $\exists \boldsymbol{R}\varphi$, where φ is a K^*-sentence for some language $K = L \cup \boldsymbol{R}$, with $\boldsymbol{R}$ a set of relation symbols not in L. If $\mathfrak{M}$ is an L-structure, $\mathfrak{M} \models_{\Sigma(L^*)} \exists \boldsymbol{R}\varphi$, for short $\mathfrak{M} \models \exists \boldsymbol{R}\varphi$, iff $\mathfrak{M} = \mathfrak{N} \restriction L$ for some K-structure $\mathfrak{N} \models_{K^*} \varphi$.

The delta logic, $\Delta(L^*)$, is defined as follows. Sentences of $\Delta(L^*)$ are ordered pairs $\langle \exists \boldsymbol{R}_1\varphi_1, \exists \boldsymbol{R}_2\varphi_2\rangle$, where $\exists \boldsymbol{R}_1\varphi_1$ and $\exists \boldsymbol{R}_2\varphi_2$ are $\Sigma(L^*)$ sentences such that for all L-structures $\mathfrak{M}$, $\mathfrak{M} \models \exists \boldsymbol{R}_1\varphi_1$ or $\mathfrak{M} \models \exists \boldsymbol{R}_2\varphi_2$, but not both. We write $\mathfrak{M} \models_{\Delta(L^*)} \langle \exists \boldsymbol{R}_1\varphi_1, \exists \boldsymbol{R}_2\varphi_2\rangle$, for short $\mathfrak{M} \models \langle \exists \boldsymbol{R}_1\varphi_1, \exists \boldsymbol{R}_2\varphi_2\rangle$, if and only if $\mathfrak{M} \models \exists \boldsymbol{R}_1\varphi_1$.

If L^* is identified with its set of EC (elementary) classes, then $\Sigma(L^*)$ is the set of PC (projective) classes, and $\Delta(L^*)$ is the set of classes which are both PC and have PC complement.

For more information on Δ-logics, see [14].

We now show that $\Delta(LQ_\kappa)$ does not satisfy interpolation for κ regular, $\kappa > \omega$. (LQ_κ is the logic having the same syntax as LQ, but $Qv\varphi$ is interpreted as "$\exists$ at least κ many v such that φ".) The case $\kappa = \omega_1$ is due to Friedman [5], his proof however used models of set theory. Barwise [1] has shown that $\Delta(LQ\omega) = L_{HH}$ on infinite structures, where HH is the set of hereditarily hyperarithmetic sets. In particular, $\Delta(LQ_\omega)$ satisfies interpolation if one disregards finite structures.

We use the following definitions.

A linear ordering $\langle L, < \rangle$ is κ-like, κ a regular cardinal, if L has cardinality κ and every proper initial segment of L has cardinality less than κ.

A linear ordering $\langle L, < \rangle$ has cofinality κ, κ a regular cardinal, if L has a cofinal subset of order type κ. Equivalently, L has a cofinal κ-like subset.

THEOREM 3.1. *$\Delta(LQ_\kappa)$ does not satisfy interpolation for κ regular, $\kappa > \omega$.*

PROOF. Let

$$K_1 = \{\mathfrak{A} = \langle A, < \rangle \colon \mathfrak{A} \text{ is a linear ordering of cofinality } \kappa\},$$

$$K_2 = \{\mathfrak{A} = \langle A, < \rangle \colon \mathfrak{A} \text{ is a linear ordering of cofinality } < \kappa\}.$$

Clearly $K_1 \cap K_2 = \emptyset$.

Since

$$\mathfrak{A} = \langle A, < \rangle \in K_1 \quad \text{iff} \quad \exists U \subset A \{\langle A, <, U\rangle \models \text{"} < \text{ is a l.o. of } A\text{"} \wedge \text{"}U \text{ is cofinal in } A\text{"} \wedge \text{"}U \text{ is } \kappa\text{-like"}\},$$

$$\mathfrak{A} = \langle A, < \rangle \in K_2 \quad \text{iff} \quad \exists U \subset A \{\langle A,, <, U\rangle \models \text{"} < \text{ is a l.o. of } A\text{"} \wedge \text{"}U \text{ is cofinal in } A\text{"} \wedge \text{"}|U| < \kappa\text{"}\},$$

it follows $K_1, K_2 \in \Sigma(LQ_\kappa) = \Sigma(\Delta(LQ_\kappa))$.

Consider now any structure $\mathfrak{A} = \langle \kappa^+, <, \boldsymbol{P}\rangle$, where $\boldsymbol{P}$ is a finite set of relations on κ^+, and let L' be the language of $\mathfrak{A}$.

Construct a sequence $\langle A_\xi \colon \xi \leqq \kappa \rangle$ of subsets of κ^+ satisfying the following conditions. $A_0 = \kappa$. Suppose A_ξ has been constructed and $|A_\xi| = \kappa$. Then $A_{\xi+1}$ is constructed so that:

(i) $|A_{\xi+1}| = \kappa$;

(ii) $\exists x \in A_{\xi+1} \; \forall y \in A_\xi \quad x > y$;

(iii) for all formulae $\theta(x, \boldsymbol{y})$ in $L'Q_\kappa$ and all $\boldsymbol{a} \in A_\xi$, if
(a) $\mathfrak{A} \models \exists x \theta(x, \boldsymbol{a})$, then $\mathfrak{A} \models \theta(b; \boldsymbol{a})$ for some $b \in A_{\xi+1}$, and if
(b) $\mathfrak{A} \models Q_\kappa x \theta(x, \boldsymbol{a})$, then $\mathfrak{A} \models \theta(b, \boldsymbol{a})$ for κ many $b \in A_{\xi+1}$.
If ξ is a limit ordinal, $A_\xi = \bigcup_{\nu<\xi} A_\nu$.

Let $\mathfrak{B} = \mathfrak{A} \restriction A_\omega$, $\mathfrak{C} = \mathfrak{A} \restriction A_\kappa$. Then $\mathfrak{B}, \mathfrak{C} <_{L'Q_\kappa} \mathfrak{A}$ by induction on formulae. Now suppose K_1, K_2 can be interpolated in $\Delta(LQ_\kappa)$. In particular, let $\langle \exists \boldsymbol{R}_1 \varphi_1, \exists \boldsymbol{R}_2 \varphi_2 \rangle$ be a $\Delta(LQ_\kappa)$ sentence such that $\mathfrak{M} \in K_1 \Rightarrow \mathfrak{M} \models \exists \boldsymbol{R}_1 \varphi_1$, and $\mathfrak{M} \in K_2 \Rightarrow \mathfrak{M} \models \exists \boldsymbol{R}_2 \varphi_2$.

If $\langle \kappa^+, < \rangle \models \exists \boldsymbol{R}_1 \varphi_1$, then $\langle \kappa^+, <, \boldsymbol{P} \rangle \models \varphi_1$ for some $\boldsymbol{P}$, hence if $\mathfrak{B}$ is constructed as before, $\mathfrak{B} \models \varphi_1$, and so $\mathfrak{M} = \mathfrak{B} \restriction < \models \exists \boldsymbol{R}_1 \varphi_1$. But $\mathfrak{M}$ has cofinality ω, i.e. $\mathfrak{M} \in K_2$, contradiction. Similarly if $\langle \kappa^+, < \rangle \models \exists \boldsymbol{R}_2 \varphi_2$, then we can construct $\mathfrak{M} = \mathfrak{C} \restriction < \models \exists \boldsymbol{R}_2 \varphi_2$ and $\mathfrak{M} \in K_1$, again a contradiction. Thus K_1, K_2 cannot be interpolated in $\Delta(LQ_\kappa)$. □

The logic $L_{\infty\lambda}$ is constructed as for $L_{\infty\omega}$, except that now quantification over sets of variables of cardinality $< \lambda$ is allowed. Furthermore, we lift any restriction on the size of the language L. All formulae of $L_{\infty\lambda}$ are to have $< \lambda$ free variables.

The following result is proved similarly to Theorem 3.1. The case $\lambda = \omega$ is due to Friedman [5], using models of set theory.

THEOREM 3.2. *For κ regular, $\Delta(L_{\kappa^{++}\omega})$ cannot be interpolated in $\Delta(L_{\infty\kappa})$. In particular, no $\Delta(L_{\infty\lambda})$ satisfy interpolation.*

PROOF. The classes K_1, K_2 of linear orderings of cofinality κ, κ^+ respectively, are disjoint PC classes in $\Delta(L_{\kappa^{++}\omega})$.

Let $\boldsymbol{R}$ be a set of relation symbols, L' the language $\boldsymbol{R} \cup \{<\}$, Φ a set of formulae of $L'_{\infty\kappa}$ closed under subformulae, $\rho = |\Phi|$, and $\sigma = \rho^\kappa$. Then as in Theorem 3.1, we can construct $\mathfrak{A}, \mathfrak{B} <_\Phi \langle \sigma^+, <, \boldsymbol{P} \rangle$ where the $\boldsymbol{P}$ interpret the $\boldsymbol{R}$, with cofinality $(\mathfrak{A}) = \kappa$, and cofinality $(\mathfrak{B}) = \kappa^+$ (also $|A| = |B| = \sigma$). The argument otherwise is similar to before. □

REMARKS. (a) In Theorem 4.2 of [15], Malitz essentially shows that

$$K = \{ \mathfrak{A} = \langle A, U_1, U_2, <_1, <_2 \rangle : \langle U_i, <_i \rangle \text{ is a well ordering, } i = 1, 2, \text{ and } \langle U_1, <_1 \rangle \cong \langle U_2, <_2 \rangle \},$$

is not an EC class in $L_{\infty\infty}$, but is an EC class in $\Delta(L_{\omega_1\omega_1})$. In particular $L_{\infty\infty} \subsetneqq \Delta(L_{\infty\infty})$.

(b) Since a set A has cardinality $\geqq \kappa$ iff A has a well-ordered subset of order type κ iff there is no well-ordering of A of order type $< \kappa$, it follows by induction on formulae of LQ_κ that $LQ_\kappa \subset \Delta(L_{\kappa^+\omega})$. Thus $\Delta(LQ_\kappa) \subset \Delta(L_{\kappa^+\omega})$ for all κ.

$\aleph_1$-LIKE ORDERINGS. Any $\aleph_1$-like linear ordering L induces an $\aleph_1$-like dense linear ordering $(1+\eta)\times L$, where η is the order type of the rationals. If L is already a dlo, then $(1+\eta)\times L \cong L$. This follows in particular from the following classification of $\aleph_1$-like dlo's due to Conway [3]. The proof is also in [8].

LEMMA 3.3. *Every $\aleph_1$-like dlo can be expressed in the form*

$$\Phi(A) = \sum_{\alpha<\omega_1} \begin{matrix} 1+\eta, & \text{if} & \alpha \in A, \\ \eta, & & \alpha \notin A, \end{matrix}$$

where $A \subset \omega_1$. For $A, B \subset \omega_1$ let us say $A \sim B$ if and only if (i) $(A\cap B)\cup(A^c \cap B^c)$ includes a closed unbounded subset of ω_1, and (ii) either $0 \in A\cap B$ or $0 \in A^c \cap B^c$. Then $\Phi(A)\cong\Phi(B)$ if and only if $A \sim B$. □

To simplify matters a little, we will assume that all $\aleph_1$-like dlo's have a first element; i.e., are of the form $\Phi(A)$ where $0 \in A \subset \omega_1$.

We investigate which logics distinguish, and which fail to distinguish, such dlo's. (By a back and forth type argument, they are all elementarily equivalent in $L_{\infty\omega_1}$.) In particular, we show that each isomorphism type is a $\Delta(L_{\omega_1\omega})$ class assuming *GCH*, but all such dlo's are elementarily equivalent in $\Delta(L_{\omega_1\omega}Q)$.

By $L_{\min}$, we will mean $1+(\eta\times\omega_1)$, i.e. $\Phi(\{0\})$.

THEOREM 3.4. *The isomorphism type of each $\aleph_1$-like dlo is a $\Sigma(L_{\omega_2\omega})$ class and a $\Pi(L_{(2^{\omega_1})^+\omega})$ class. The isomorphism type of $L_{\min}$ is a $\Sigma(LQ)$ class, and the class of L's $\not\cong L_{\min}$ is a $\Pi(LQ)$ class.*

PROOF. For each $\xi<\omega_1$, there is an $L_{\omega_1\omega}$ formula $\varphi_\xi(v)$ such that for $\langle L,<\rangle$ a l.o. and $X\subset L$, $\langle L,<,X\rangle \models \varphi_\xi(a)$ if and only if $\{b : b<a \wedge b\in X\}$ has order type ξ.

Let $\mathscr{F}=\{U : U\subset\{\xi:\xi<\omega_1 \wedge \lim(\xi)\}\}$. For $U\in\mathscr{F}$, let θ_U be the following $L_{\omega_2\omega}$ sentence in the language of $\langle L,<,X\rangle$, where X is a unary relation.

"$\langle L,<\rangle$ is a dlo with a first element"

$\wedge$ "X is cofinal in $\langle L,<\rangle$"

$\wedge\, \forall x(x\notin X \rightarrow \exists y<x\,\forall z(y<z<x\rightarrow z\notin X)$

(i.e., X is closed relative to $\langle L,<\rangle$)

$$\wedge \bigwedge_{\xi<\omega_1} \exists x \varphi_\xi(x) \wedge \forall x \in X \bigvee_{\xi<\omega_1} \varphi_\xi(x)$$

$$\wedge \bigwedge_{\xi \in U} \exists x \left[\varphi_\xi(x) \wedge \forall y < x \exists z (y < z < x \wedge \bigvee_{\nu<\xi} \varphi_\nu(z)) \right]$$

(i.e., $\varphi_\xi(x) \Rightarrow x$ is the limit of $\{y : \varphi_\nu(y) \wedge \nu < \xi\}$)

$$\wedge \bigwedge_{\substack{\xi \notin U \\ \lim(\xi) \\ \xi<\omega_1}} \exists x \left[\varphi_\xi(x) \wedge \exists y < x \forall z (y < z < x \rightarrow \bigwedge_{\nu<\xi} \neg \varphi_\nu(z)) \right]$$

(i.e., $\varphi_\xi(x) \Rightarrow x$ is not the limit of $\{y : \varphi_\nu(y) \wedge \nu < \xi\}$, and so by the third conjunct, this set does not have a limit point in L).

From Lemma 3.3, if $\langle L, < \rangle$ is an $\aleph_1$-like dlo, $0 \in A \subset \omega_1$, and $U = \{\xi : \xi \in A \wedge \lim \xi\}$, then $\langle L, < \rangle \cong \Phi(A)$ iff $\langle L, < \rangle \models \exists X \theta_U$ iff

$$\langle L, < \rangle \models \forall X \bigwedge_{\substack{\nu \in \mathcal{F} \\ V \neq U}} \neg \theta_V.$$

Notice that $|\mathcal{F}| = 2^{\omega_1}$.

But we can express in both $\Sigma(L_{\omega_2\omega})$ and $\Pi(L_{\omega_3\omega})$, that $\langle L, < \rangle$ is an $\aleph_1$-like dlo. (Basically, this is possible because in both $\Sigma(L_{\kappa^+\omega})$ and $\Pi(L_{\kappa^{++}\omega})$, one can express the notion of having cardinality λ for each $\lambda \leqq \kappa$.) Thus we have proved the first sentence of the theorem.

$\langle L, < \rangle \cong L_{\min}$ if and only if $\langle L, < \rangle$ is an $\aleph_1$-like dlo with a cofinal subset X, such that no member of L is a limit point of X. Hence the isomorphism type of $L_{\min}$ is a $\Sigma(LQ)$ class. Finally, the class of $\aleph_1$-like dlo's is an LQ class, and so the class of L's $\not\cong L_{\min}$ is a $\Pi(LQ)$ class. □

THEOREM 3.5. *All $\aleph_1$-like dlo's not isomorphic to $L_{\min}$ are $\Sigma(L_{\omega_1\omega}Q)$ elementarily equivalent. If an $\aleph_1$-like dlo belongs to some $\Sigma(L_{\omega_1\omega}Q)$ class, then so does $L_{\min}$ (but not conversely by the previous theorem). In particular, all $\aleph_1$-like dlo's are $\Delta(L_{\omega_1\omega}Q)$ elementarily equivalent.*

PROOF. The proof uses models of set theory. Let $\mathcal{L} = \langle L, < \rangle = \Phi(K) \models \exists \boldsymbol{R}\varphi$, where $\exists \boldsymbol{R}\varphi \in \Sigma(L_{\omega_1\omega}Q)$, and $\mathcal{L} \not\cong L_{\min}$. Then K is stationary. We may assume $TC(\{\exists \boldsymbol{R}\varphi\}) \in HC$, where HC is the set of hereditarily countable sets.

Let $\mathcal{L}' = \Phi(K')$ be any $\aleph_1$-like dlo. We will show $\mathcal{L}' \models \exists \boldsymbol{R}\varphi$.

Take $A \prec V$ such that $|A| = \aleph_0$, $\mathcal{L}$, $K \in A$, $TC(\{\exists \boldsymbol{R}\varphi\}) \in A$. Using Theorem

1.2, construct a sequence $\mathfrak{A}_\xi = \langle A_\xi, E_\xi \rangle$, $\xi < \omega_1$, such that: (i) $\mathfrak{A}_0 = \langle A, \in \rangle$; (ii) $\mathfrak{A}_{\xi+1} > \mathfrak{A}_\xi$, $|A_{\xi+1}| = \aleph_0$, all ordinals below $\omega_1^{\mathfrak{A}_\xi}$ remain fixed, $\omega_1^{\mathfrak{A}_\xi}$ is enlarged, $\omega_1^{\mathfrak{A}_{\xi+1}}$ contains a least new ordinal a_ξ if $\xi \in K'$ and in this case $a_\xi \in (K^{\mathfrak{A}_{\xi+1}})_{E_{\xi+1}}$, $\omega_1^{\mathfrak{A}_{\xi+1}}$ contains no least new ordinal if $\xi \notin K'$; and (iii) $\mathfrak{A}_\xi = \bigcup_{\nu<\xi} \mathfrak{A}_\nu$ if ξ is a limit ordinal. Let $\mathfrak{B} = \langle B, F \rangle = \bigcup_{\xi<\omega_1} \mathfrak{A}_\xi$.

Now $\mathcal{L} \models \exists R\varphi$ in the real world and hence in $\mathfrak{B}$. So in $\mathfrak{B}$, $\langle \mathcal{L}, \boldsymbol{P} \rangle \models \varphi$ for some $\boldsymbol{P}$. Therefore $\langle \mathcal{L}, \boldsymbol{P} \rangle_F \models \varphi$ in the real world, since $\varphi \in \in \mathfrak{B}$, $|(\omega^{\mathfrak{B}})_F| = \aleph_0$, and $|(\omega_1^{\mathfrak{B}})_F| = \aleph_1$. Therefore $\mathcal{L}_F \models \exists \boldsymbol{R}\varphi$.

In $\mathfrak{B}$, $\mathcal{L}$ is an $\aleph_1$-like dlo, so $\mathcal{L}_F$ is an $\aleph_1$-like dlo in the real world, as all these notions are absolute. Also, the notion of having the order type of the rationals is absolute. We show $\mathcal{L}_F \cong \mathcal{L}'$.

We have

$$\mathcal{L}_F = \sum_{k \in (\omega_1^{\mathfrak{B}})_F} \begin{array}{lll} 1+\eta, & \text{if} & k \in K_F \\ \eta, & \text{if} & k \notin K_F \end{array}$$

$$= \sum_{\alpha \in \omega_1} \left\{ \sum_{k \in (\omega_1^{\mathfrak{A}_{\xi+1}})F - (\omega_1^{\mathfrak{A}_\xi})F} \begin{array}{lll} 1+\eta, & \text{if} & k \in K_F \\ \eta, & \text{if} & k \notin K_F \end{array} \right\}$$

$$= \sum_{\alpha \in \omega_1} \begin{array}{lll} 1+\eta, & \text{if} & \alpha \in K' \\ \eta, & \text{if} & \alpha \notin K' \end{array}$$

(since $\{-\}$ is a countable dlo with no last element, and having a first element if and only if $\alpha \in K'$)

$$= \Phi(K').$$

□

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